Exercise 3

Find the series solution for the following homogeneous second order ODEs:

$$u'' - (1+x)u' + u = 0$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Now we substitute these series into the ODE.

$$u'' - (1+x)u' + u = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - (1+x)\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$. In addition, the second series is zero for $n = 0$, so we can start the sum from $n = 1$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Since we want to combine the series, we want the first two series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$, and we can start the second at $n = 0$ as long as we replace n with $n + 1$.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

The point of doing this is so that x^n is present in each term so we can combine the series.

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} x^n - (n+1)a_{n+1} x^n - n a_n x^n + a_n x^n] = 0$$

Factor the left side.

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n(n-1)]x^n = 0$$

Thus,

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n(n-1) = 0$$
$$a_{n+2} = \frac{(n+1)a_{n+1} + (n-1)a_n}{(n+2)(n+1)}.$$

Now that we know the recurrence relation, we can determine the coefficients.

$$\begin{aligned}
n = 0: &\qquad a_2 = \frac{-a_0 + a_1}{2} \\
n = 1: &\qquad a_3 = \frac{a_2}{3} = \frac{-a_0 + a_1}{6} \\
n = 2: &\qquad a_4 = \frac{3a_3 + a_2}{12} = \frac{-a_0 + a_1}{12} \\
n = 3: &\qquad a_5 = \frac{4a_4 + 2a_3}{20} = \frac{-a_0 + a_1}{30} \\
n = 4: &\qquad a_6 = \frac{5a_5 + 3a_4}{30} = \frac{-a_0 + a_1}{72} \\
n = 5: &\qquad a_7 = \frac{6a_6 + 4a_5}{42} = \frac{13(-a_0 + a_1)}{2520} \\
\vdots &\qquad \vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x) = a_0 &\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{30}x^5 - \frac{1}{72}x^6 - \frac{13}{2520}x^7 - \cdots\right) \\
&+ a_1\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \frac{1}{72}x^6 + \frac{13}{2520}x^7 + \cdots\right),
\end{aligned}$$

where a_0 and a_1 are arbitrary constants.